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A BODY-BOUND NAVIGATION SYSTEM

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Analysis and Design of a Capsule Landing System
and Surface Vehicle Control System for
Mars Exploration

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ABSTRACT

A requirement for an unmanned Martian roving vehicle is an autonomous onboard navigation system. One necessity of such a system is to erect a reference coordinate frame and relate any subsequent motion to this frame.

The instrumentation decided upon to erect a local vertical, ergo a reference coordinate system, was two body mounted two-degrees-of-freedom gyroscopes. This will supply three orthogonal reference axes with one axis redundant. The outputs of the gyroscopes (i.e. the gimbal angles and the gimbal angular velocities), when used in conjunction with the components of acceleration of the rover, are a set of sufficient conditions to completely specify the vehicle's Euler angle rotation and translation from the initial reference frame.

This report develops the necessary equations of motion of a two-degrees-of-freedom gyroscope where large angular excursions occur, the relationships between the various coordinate frames employed, and the translational equations of motion of the rover itself.

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Reference Frames

(All of the following are orthonormal right-hand systems.)

$\bar{b}_k$	vehicle principal axes: $\bar{b}_1$ roll axis; $\bar{b}_2$ pitch axis; $\bar{b}_3$ yaw axis.
$\bar{e}_k$	intermediate axes system (used to rotate $\bar{s}$ frame into $\bar{b}$ frame, $\bar{e}_3 = \bar{s}_3$)
$\bar{f}_k$	intermediate axes system (used to rotate $\bar{s}$ frame into $\bar{b}$ frame, $\bar{f}_2 = \bar{e}_2$)
$\bar{g}_{i_k}$	inner gimbal axes
$\bar{g}_{o_k}$	outer gimbal axes
$\bar{I}_k$	inertial axes (fixed at center of planet)
$\bar{p}_k$	planet axes
$\bar{r}_k$	rotor axes
$\bar{s}_k$	planet's surface axes
$\bar{y}_k$	intermediate axes system (used to rotate $\bar{p}$ frame into $\bar{s}$ frame, $\bar{y}_3 = \bar{p}_3$)

System Variables

(All angles correspond to right-hand rotation.)

$\bar{a}$	inertial acceleration of the vehicle
C_{i_Q}, C_{o_Q}	damping coefficient of the inner and outer gimbal, respectively for the Q gyro
$\bar{F}_g$	force due to gravity
$\bar{F}_r$	reaction force of the planet on the vehicle
$\bar{F}_v$	internal force generated by the vehicle (assumed to act in $\bar{b}_1$ direction)
g	planet's gravitational constant
h_Q	rotor momentum about spin axis of the Q gyro

I_{jr}	moment of inertia of gyro rotor (where j is the direction of the spin axis in the $\bar{r}$ frame)
k_{i_Q}, k_{o_Q}	spring constant of inner and outer gimbal, respectively for the Q gyro
M	total mass of the vehicle
$\bar{R}$	vector from center of planet to the vehicle
$\bar{V}$	inertial velocity of the vehicle
x,y,z	coordinates of the vehicle in the $\bar{p}$ reference frame
ψ_b, θ_b, ϕ_b	Euler angles orienting the $\bar{b}$ reference frame with respect to the $\bar{s}$ frame (Order of rotation is as shown.)
ψ_L, θ_L	Euler angles orienting the $\bar{s}$ reference frame with respect to the $\bar{p}$ frame (Order of rotation is as shown.)
θ_{i_Q}	angle orienting the $\bar{g}_i$ reference frame with respect to the $\bar{g}_o$ frame of the Q gyro
ϕ_{o_Q}	angle orienting the $\bar{g}_o$ reference frame with respect to the $\bar{b}$ frame of the Q gyro
ψ_{r_Q}	angle orienting the $\bar{r}$ reference frame with respect to the $\bar{g}_i$ frame of the Q gyro
ψ_k	angle orienting the $\bar{p}$ reference frame with respect to the $\bar{I}$ frame
$\bar{\omega}_j$	inertial angular velocity of the j reference frame

Subscripts

b	vehicle principal axes
g_{i_Q}	inner gimbal axes of the Q gyro
g_{o_Q}	outer gimbal axes of the Q gyro
k = 1, 2, 3	axes system components

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I. INTRODUCTION

Before the turn of the century the United States plans to land an autonomously controlled unmanned roving vehicle on the planet Mars. This report is concerned with the study of a navigation system for a typical vehicle. Such a system must be capable of acting in either a passive or active mode. To perform in the passive mode, a self contained navigation system must be able to supply changes in translational and Eulerian parameters from an initial reference position. On the other hand, the rover may require a specific orientation in order to perform a mission; an active navigation scheme is, therefore, necessitated.

A stable platform navigation system was not considered due to size, weight, and performance restrictions; therefore, attention was turned to a body mounted gyroscopic system. The primary obstacle in the development of such a system is that large angular excursions of the gimbals are encountered.

The authors formulated two questions upon which to base their investigations; first, could gyroscopic outputs be related to system parameters; and second, could a physically realizable system be obtained? Reference 3 investigates the feasibility of a body-bound navigation system incorporating single-degree-of-freedom gyroscopes; therefore, the authors turned their efforts towards a system employing two-degrees-of-freedom gyroscopes, the report of which is presented herein.

Section II develops the relationships between the outputs of the gyroscopes (i.e. the gimbal angles and the gimbal angular velocities), the measured sum of the torques about the output axes of the gimbals, and the vehicle angular velocity ($\bar{\omega}_b$). Section III uses the results of Section II in conjunction with the integrated components of the rover's measured acceleration along its principal axes to obtain expressions for the translational and Eulerian movement of the craft. The final section (IV) of the report presents the conclusions and recommendations of the authors with regard to future endeavors in the design of a body-bound navigation system.

II. ANALYSIS OF BODY-BOUND TWO-DEGREES-OF-FREEDOM GYROSCOPES AS AN INERTIAL BODY RATE SENSOR

Consider two two-degree-of-freedom gyroscopes mounted at the center of gravity of the martian land rover. Since the vehicle is always in contact with the Martian surface, each gyro's momentum does not affect any dynamical modes of the craft. The gyros are, however, affected by any changes in the inertial angular velocity of the rover.

Appendix A gives the derivation of the equations of motion of a two-degrees-of-freedom gyro in a 1, 2, 3 configuration. (The 1, 2, 3 designation refers to the directions of the outer gimbal output axis, the inner gimbal output axis, the inner gimbal output axis, and the spin axis of the rotor, respectively). These equations are:

$$-h\omega_{g_{i_1}}\bar{g}_{i_2} = \Sigma \text{ external } \mathcal{T} \cdot \bar{g}_{i_2}\,, \tag{1}$$

and

$$h\,c\,\theta_i(\dot{\theta}_i + \omega_{g_{o_2}})\,\bar{g}_{o_1} = \Sigma \text{ external } \mathcal{T} \cdot \bar{g}_{o_1}\,, \tag{2}$$

where c = cosine. The term on the left side of the equals sign in equations (1) and (2) represents the time derivatives of the moment of momentum for the inner gimbal dynamics and outer gimbal dynamics, respectively, about the output axis of the corresponding gimbal.

Assuming all forces but the external torques on the gimbals (a viscous damping force and a spring restraining force) are negligible, the right hand side of equations (1) and (2) expand into

$$\Sigma \text{ external } \mathcal{T} = K_j\alpha + C_j\dot{\alpha}\,, \tag{3}$$

where the subscript j indexes either the inner or outer gimbal, and the parameter α refers to the measured gimbal angle variable under consideration: θ_i for the inner gimbal or ϕ_o for the outer gimbal.

Designating the 1,2,3 gyro configuration by the subscript A, and combining equation (1) with (3), and equation (2) with (3) yields

$$-h_A \omega_{g_{i_{1_A}}} \bar{g}_{i_2} = K_{i_A} \theta_{i_A} + C_{i_A} \dot{\theta}_{i_A} , \qquad (4)$$

and

$$h_A c\theta_{i_A} (\dot{\theta}_{i_A} + \omega_{g_{o_{2_A}}}) \bar{g}_{o_1} = K_{o_A} \phi_{o_A} + C_{o_A} \dot{\phi}_{o_A} , \qquad (5)$$

respectively.

Following a similar analysis for a gyro in the 2,3,1 configuration (see Appendix A), designated by the subscript B, the two gyro defining equations are found to be

$$-h_B \omega_{g_{i_{2_B}}} \bar{g}_{i_3} = K_{i_B} \theta_{i_B} + C_{i_B} \dot{\theta}_{i_B} , \qquad (6)$$

and

$$h_B c\theta_{i_B} (\dot{\theta}_{i_B} + \omega_{g_{o_{3_B}}}) \bar{g}_{o_2} = K_{o_B} \phi_{o_B} + C_{o_B} \dot{\phi}_{o_B} . \qquad (7)$$

Assuming each gyro's outputs to be its gimbal angles and its gimbal angle rates, equations (4), (5), (6), and (7) contain four unknowns; namely, $\omega_{g_{i_{1_A}}}$, $\omega_{g_{i_{2_B}}}$, $\omega_{g_{o_{2_A}}}$, and $\omega_{g_{o_{3_B}}}$, respectively, for which a solution can readily be found. The quantities θ_{i_Q}, $\dot{\theta}_{i_Q}$, and ϕ_{o_Q}, $\dot{\phi}_{o_Q}$ (where the Q references either the A or B gyro) are measured directly. The remaining constants are determined by the specifications of the exact gyroscope used.

Equation (A13) in Appendix A can be expanded to yield the above four variables as functions of the unknowns ω_{b_1}, ω_{b_2}, and ω_{b_3}, with respect to the gyro configurations, A and B,

$$\omega_{g_{i_{1_A}}} = (\omega_{b_1} + \dot{\phi}_{c_A})\, c\,\theta_{i_A} + (\omega_{b_2} + \dot{\theta}_{i_A} c\,\phi_{o_A}) \cdot s\,\theta_{i_A} s\,\phi_{o_A} - (\omega_{b_3} + \dot{\theta}_{i_A} s\,\phi_{o_A}) s\,\theta_{i_A} c\,\phi_{o_A}, \quad (8)$$

$$\omega_{g_{o_{2_A}}} = \omega_{b_2} c\,\phi_{o_A} + \omega_{b_3} s\,\phi_{o_A}, \quad (9)$$

$$\omega_{g_{i_{2_B}}} = -(\omega_{b_1} + \dot{\theta}_{i_B} s\,\phi_{o_B}) s\,\theta_{i_B} c\,\phi_{o_B} + (\omega_{b_2} + \dot{\phi}_{o_B})\, c\,\theta_{i_B} + (\omega_{b_3} + \dot{\theta}_{i_B} c\,\phi_{o_B}) s\,\theta_{i_B} s\,\phi_{o_B}, \quad (10)$$

$$\omega_{g_{o_{3_B}}} = \omega_{b_1} s\,\phi_{o_B} + \omega_{b_3} c\,\phi_{o_B}, \quad (11)$$

where s = sine.

Equations (8), (9), (10), and (11) contain only three unknowns, so that both solution and verification are possible. The inertial angular velocity components of the vehicle along its principal axes are found to be:

$$\omega_{b_1} = \left[\frac{1}{c\,\theta_{i_A}} \quad \omega_{g_{i_{1_A}}} - \phi_{c_A} c\,\theta_{i_A} - s\,\theta_{i_A} s\,\phi_{o_A}\left(\frac{E\, s\,\phi_{o_A} - W \omega_{g_{o_{2A}}}}{\Delta}\right) - \dot{\theta}_{i_A} c\,\phi_{c_A} s\,\theta_{i_A} s\,\phi_{o_A} + s\,\theta_{i_A} c\,\phi_{o_A}\left(\frac{N\omega_{g_{o_{2A}}} - E\,c\,\phi_{o_A}}{\Delta}\right) + \dot{\theta}_{i_A} s\,\phi_{o_A} s\,\theta_{i_A} c\,\phi_{o_A}\right], \quad (12)$$

$$\omega_{b_2} = \frac{E\, s\,\phi_{o_A} - W\omega_{g_{o_{2A}}}}{\Delta}, \quad (13)$$

and

$$\omega_{b_3} = \frac{N\,\omega_{go_{3A}} - E\,c\phi_{o_A}}{\Delta} \tag{14}$$

where

$$\begin{aligned} E = {} & \omega_{gi_{1A}}\, s\theta_{i_B}\, c\phi_{o_B} + \omega_{gi_{2B}}\, c\theta_{i_A} - \dot{\phi}_{o_A}\, c\theta_{i_A}\, s\theta_{i_B}\, c\phi_{o_B} \\ & - \dot{\theta}_{i_A}\, c\phi_{o_A}\, s\theta_{i_A}\, s\phi_{o_A}\, s\theta_{i_B}\, c\phi_{o_B} + \dot{\theta}_{i_A}\, s\phi_{o_A} \big(s\theta_{i_A} \cdot \\ & c\phi_{o_A}\, s\theta_{i_B}\, c\phi_{o_B} \big) + \dot{\theta}_{i_B}\, s\phi_{o_B}\, s\theta_{i_B}\, c\phi_{o_B}\, c\theta_{i_A} - \dot{\phi}_{o_B} \cdot \\ & c\theta_{i_B}\, c\theta_{i_A} - \dot{\theta}_{i_B}\, c\phi_{o_B}\, s\theta_{i_B}\, s\phi_{o_B}\, c\theta_{i_A} \,, \end{aligned}$$

$$W = s\theta_{i_B}\, s\phi_{o_B}\, c\theta_{i_A} - s\theta_{i_A}\, c\phi_{o_A}\, s\theta_{i_B}\, c\phi_{o_B} \,,$$

$$N = s\theta_{i_A}\, s\phi_{o_A}\, s\theta_{i_B}\, c\phi_{o_B} + c\theta_{i_B}\, c\theta_{i_A} \,,$$

and

$$\Delta = N\, s\phi_{o_A} - W\, c\phi_{o_A} \,.$$

III. DESIGN OF A NAVIGATION SYSTEM INCORPORATING TWO BODY-BOUND TWO-DEGREES-OF-FREEDOM GYROSCOPES

The navigation system presented in this report has been investigated in terms of its acting as a passive system. This implies that, given the initial attitude and position of the rover, the body-bound system will continuously update this data as the rover traverses the planet. It is a simple matter, though, to show that the final results are applicable to both active and passive modes of operation of the system.

Figure 3 shows the location of the rover in the $\bar{p}_1$, $\bar{p}_2$, $\bar{p}_3$ reference frame (centered in the planet, and rotating with it) as denoted by x, y, and z, respectively. It is, however, advantageous to describe the vehicle's position in terms of spherical coordinates,

$$x = R\,s\theta_L\,c\psi_L \quad , \tag{15}$$

$$y = R\,s\theta_L\,s\psi_L \quad , \tag{16}$$

and

$$z = R\,c\theta_L \quad . \tag{17}$$

where R equals the radius of the planet and is assumed constant. Equations (15), (16), and (17) can be used to solve for ψ_L, $\dot{\psi}_L$, θ_L, and $\dot{\theta}_L$ in terms of the location and velocity of the vehicle. Therefore,

$$\theta_L = c^{-1}\left(\frac{z}{R}\right) , \qquad (18)$$

$$\dot{\theta}_L = - \frac{\dot{z}}{R\sqrt{1-\left(\frac{z}{R}\right)^2}} , \qquad (19)$$

$$\psi_L = \tan^{-1}\left(\frac{y}{x}\right) , \qquad (20)$$

and

$$\dot{\psi}_L = \frac{x\dot{y} - y\dot{x}}{R^2} . \qquad (21)$$

The velocity components in the $\bar{p}$ frame, $\dot{x}$, $\dot{y}$, and $\dot{z}$; are determined by using the integrated outputs of the onboard accelerometer sensors in conjunction with equation (C18) from Appendix C. These sensors measure the components of the rover's acceleration in the $\bar{b}_1$ and $\bar{b}_2$ directions. Therefore, the two expressions generated by equation (C18), when used with equation (C3), are

$$\dot{x} = f\left(x, y, z, V_{b_1}, V_{b_2}, \theta_b, \psi_b, \phi_b, \theta_L, \psi_L\right) , \qquad (22)$$

$$\dot{y} = g\left(x, y, z, V_{b_1}, V_{b_2}, \theta_b, \psi_b, \phi_b, \theta_L, \psi_L\right) , \qquad (23)$$

and

$$\dot{z} = h\left(x, y, z, V_{b_1}, V_{b_2}, \theta_b, \psi_b, \phi_b, \theta_L, \psi_L\right) , \qquad (24)$$

where the functional relationships, f, g, and h, are not explicitly shown. Equation (C4) is used to verify that all velocity and acceleration values agree.

The final equation used in determining the attitude and position of the vehicle is the translation equation, (C20), as derived in Appendix C. This equation expands into three expressions. These three relationships when used in conjunction with equations (22), (23), and (24); along with the three acceleration relationships, are a set of nine equations in nine unknowns, which can be solved to uniquely determine the rover's attitude and location.

IV. CONCLUSION

This report demonstrates that for the first time an unambiguous representation of the vehicle's orientation can be found relative to some reference system. Using the equations of motion of a two-degree-of-freedom gyroscope the vehicular angular velocity was found as a single valued function of the measured gimbal angles and the time derivatives of these angles. This result was then used in conjunction with the acceleration (as determined by onboard accelerometers) and the translation of the vehicle to obtain expressions for the components of the vehicle's velocity. Upon expressing all quantities in terms of the inertial reference coordinate frame, a set of nine equations in nine unknowns results.

A simulation of these equations on an IBM-360/50 yields the result that a unique set of the vehicle's translational, and Eulerian rotational parameters can indeed be found.

Further work in this area should be directed along the following two lines: an error analysis, and a breadboarding of the unit. These two investigations would transfer the unit from the realm of mathematical feasibility into a realizable, functioning unit.

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APPENDIX A

Derivation of Equations of Motion of a Two-Degrees-of-Freedom Gyroscope

The relationships between the orthonormal coordinate systems based in the rotor, the inner gimbal, the outer gimbal, and the body (as shown in Figure 1) can be used to define the dynamics of the gyroscope.

Consider first the rotation of the $\bar{r}$ coordinate system, fixed in the rotor, with respect to the $\bar{g}_i$ frame, fixed in the inner gimbal. This rotation is described by the matrix equation

$$\begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix} = \begin{bmatrix} c\psi_r & s\psi_r & 0 \\ -s\psi_r & c\psi_r & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{g}_{i_1} \\ \bar{g}_{i_2} \\ \bar{g}_{i_3} \end{bmatrix}, \qquad \text{(A1)}$$

where s = sine and c = cosine. The velocity relationship between the $\bar{r}$ and $\bar{g}_i$ frames is

$$\bar{\omega}_r = \bar{\omega}_{g_i} + \dot{\psi}_r \bar{g}_{i_3}, \qquad \text{(A2)}$$

where ω_j is the angular velocity of the j coordinate system with respect to inertial space, and $\dot{\psi}_r \bar{g}_{i_3}$ is the relative angular velocity of the two frames.

There are two additional dynamical modes of the gyroscope:

a) the rotation of the inner gimbal with respect to the outer gimbal ($\bar{g}_o$ coordinate system); and b) the motion of the outer gimbal with respect to the body ($\bar{b}$ frame). Each mode can be described by two equations of the form of (A1) and (A2): a matrix frame transformation and the inter-frame velocity relationship.

The equations describing the motion of mode a) are

$$\begin{bmatrix} \bar{g}_{i_1} \\ \bar{g}_{i_2} \\ \bar{g}_{i_3} \end{bmatrix} = \begin{bmatrix} c\theta_i & 0 & -s\theta_i \\ 0 & 1 & 0 \\ s\theta_i & 0 & c\theta_i \end{bmatrix} \begin{bmatrix} \bar{g}_{o_1} \\ \bar{g}_{o_2} \\ \bar{g}_{o_3} \end{bmatrix} , \tag{A3}$$

and

$$\bar{\omega}_{g_i} = \bar{\omega}_{g_o} + \dot{\theta}_i \bar{g}_{o_2} . \tag{A4}$$

And the mode b) relationships are

$$\begin{bmatrix} \bar{g}_{o_1} \\ \bar{g}_{o_2} \\ \bar{g}_{o_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi_o & s\phi_o \\ 0 & -s\phi_o & c\phi_o \end{bmatrix} \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \bar{b}_3 \end{bmatrix} , \tag{A5}$$

and

$$\bar{\omega}_{g_o} = \bar{\omega}_b + \dot{\phi}_o \bar{b}_1 . \tag{A6}$$

Expanding equations (A2), (A4), and (A6) gives

$$\begin{bmatrix} \bar{r}_1 & \bar{r}_2 & \bar{r}_3 \end{bmatrix} \begin{bmatrix} \omega_{r_1} \\ \omega_{r_2} \\ \omega_{r_3} \end{bmatrix} = \begin{bmatrix} \bar{g}_{i_1} & \bar{g}_{i_2} & \bar{g}_{i_3} \end{bmatrix} \begin{bmatrix} \omega_{g_{i_1}} \\ \omega_{g_{i_2}} \\ \omega_{g_{i_3}} \end{bmatrix} + \begin{bmatrix} \bar{g}_{i_1} & \bar{g}_{i_2} & \bar{g}_{i_3} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\psi}_r \end{bmatrix} , \tag{A7}$$

$$\begin{bmatrix}\bar{g}_{i_1} & \bar{g}_{i_2} & \bar{g}_{i_3}\end{bmatrix}\begin{bmatrix}\omega_{g_{i_1}}\\ \omega_{g_{i_2}}\\ \omega_{g_{i_3}}\end{bmatrix} = \begin{bmatrix}\bar{g}_{o_1} & \bar{g}_{o_2} & \bar{g}_{o_3}\end{bmatrix}\begin{bmatrix}\omega_{g_{o_1}}\\ \omega_{g_{o_2}}\\ \omega_{g_{o_3}}\end{bmatrix}$$

$$+ \begin{bmatrix}\bar{g}_{o_1} & \bar{g}_{o_2} & \bar{g}_{o_3}\end{bmatrix}\begin{bmatrix}0\\ \dot{\theta}_i\\ 0\end{bmatrix}, \qquad \text{(A8)}$$

and

$$\begin{bmatrix}\bar{g}_{o_1} & \bar{g}_{o_2} & \bar{g}_{o_3}\end{bmatrix}\begin{bmatrix}\omega_{g_{o_1}}\\ \omega_{g_{o_2}}\\ \omega_{g_{o_3}}\end{bmatrix} = \begin{bmatrix}\bar{b}_1 & \bar{b}_2 & \bar{b}_3\end{bmatrix}\begin{bmatrix}\omega_{b_1}\\ \omega_{b_2}\\ \omega_{b_3}\end{bmatrix}$$

$$+ \begin{bmatrix}\bar{b}_1 & \bar{b}_2 & \bar{b}_3\end{bmatrix}\begin{bmatrix}\dot{\phi}_o\\ 0\\ 0\end{bmatrix}. \qquad \text{(A9)}$$

It is desirable to express $\bar{\omega}_{g_i}$ and $\bar{\omega}_{g_o}$ as functions of $\bar{\omega}_b$ to facilitate later work. The first step is to express all inertial angular velocities in the $\bar{r}$ frame. Substituting equations (A8) and (A3) into equation (A7) gives

$$\begin{bmatrix}\bar{r}_1 & \bar{r}_2 & \bar{r}_3\end{bmatrix}\begin{bmatrix}\omega_{r_1}\\ \omega_{r_2}\\ \omega_{r_3}\end{bmatrix} = \begin{bmatrix}\bar{g}_{o_1} & \bar{g}_{o_2} & \bar{g}_{o_3}\end{bmatrix}\begin{bmatrix}\omega_{g_{o_1}}\\ \omega_{g_{o_2}}\\ \omega_{g_{o_3}}\end{bmatrix}$$

$$+ \begin{bmatrix} \bar{g}_{o_1} & \bar{g}_{o_2} & \bar{g}_{o_3} \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\theta}_i \\ 0 \end{bmatrix} + \begin{bmatrix} \bar{g}_{o_1} & \bar{g}_{o_2} & \bar{g}_{o_3} \end{bmatrix} \begin{bmatrix} c\theta_i & 0 & s\theta_i \\ 0 & 1 & 0 \\ -s\theta_i & 0 & c\theta_i \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\psi}_r \end{bmatrix} . \quad \text{(A10)}$$

Now substituting equations (A5) and (A9) into equation (A10) produces

$$\begin{bmatrix} \bar{r}_1 & \bar{r}_2 & \bar{r}_3 \end{bmatrix} \begin{bmatrix} \omega_{r_1} \\ \omega_{r_2} \\ \omega_{r_3} \end{bmatrix} = \begin{bmatrix} \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \end{bmatrix} \begin{bmatrix} \omega_{b_1} \\ \omega_{b_2} \\ \omega_{b_3} \end{bmatrix} + \begin{bmatrix} \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \end{bmatrix} \begin{bmatrix} \dot{\phi}_o \\ 0 \\ 0 \end{bmatrix}$$

$$+ \begin{bmatrix} \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi_o & -s\phi_o \\ 0 & s\phi_o & c\phi_o \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\theta}_i \\ 0 \end{bmatrix}$$

$$+ \begin{bmatrix} \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi_o & -s\phi_o \\ 0 & s\phi_o & c\phi_o \end{bmatrix} \begin{bmatrix} c\theta_i & 0 & s\theta_i \\ 0 & 1 & 0 \\ -s\theta_i & 0 & c\theta_i \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\psi}_r \end{bmatrix} . \quad \text{(A11)}$$

All terms containing $\dot{\psi}_r$ in equations (A7) and (A11) will be equal when they are expressed in the same coordinate system. Upon equating (A7) and (A11) these terms would be eliminated. Thefore, $\dot{\psi}_r$ can be set equal to zero immediately to facilitate the mathematics. Hence,

$$\begin{bmatrix} \bar{g}_{i_1} & \bar{g}_{i_2} & \bar{g}_{i_3} \end{bmatrix} \begin{bmatrix} \omega_{g_{i_1}} \\ \omega_{g_{i_2}} \\ \omega_{g_{i_3}} \end{bmatrix} = \begin{bmatrix} \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \end{bmatrix} \begin{bmatrix} \omega_{b_1} + \dot{\phi}_o \\ \omega_{b_2} + \dot{\theta}_i c\phi_o \\ \omega_{b_3} + \dot{\theta}_i s\phi_o \end{bmatrix} . \quad \text{(A12)}$$

Solving for $\bar{g}_i$ as a function of $\bar{b}$ in equations (A3) and (A5), and applying this result to equation (A12) yields

$$\begin{bmatrix} \omega_{g_{i_1}} \\ \omega_{g_{i_2}} \\ \omega_{g_{i_3}} \end{bmatrix} = \begin{bmatrix} c\theta_i & s\theta_i s\phi_o & -s\theta_i c\phi_o \\ 0 & c\phi_o & s\phi_o \\ s\theta_i & -c\theta_i s\phi_o & c\theta_i c\phi_o \end{bmatrix}$$

$$\cdot \begin{bmatrix} \omega_{b_1} + \dot{\phi}_o \\ \omega_{b_2} + \dot{\theta}_i c\phi_o \\ \omega_{b_3} + \dot{\theta}_i s\phi_o \end{bmatrix} , \quad \text{(A13)}$$

where the vectors have been eliminated.

Secondly, to obtain the ω_{g_o} equations (A10) and (A11) are equated with $\dot{\psi}_r \equiv 0$ and $\dot{\theta}_i \equiv 0$.

$$\begin{bmatrix} \bar{g}_{o_1} & \bar{g}_{o_2} & \bar{g}_{o_3} \end{bmatrix} \begin{bmatrix} \omega_{g_{o_1}} \\ \omega_{g_{o_2}} \\ \omega_{g_{o_3}} \end{bmatrix} = \begin{bmatrix} \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \end{bmatrix} \begin{bmatrix} \omega_{b_1} + \dot{\theta}_o \\ \omega_{b_2} \\ \omega_{b_3} \end{bmatrix}, \quad \text{(A14)}$$

Using equation (A5) to transform (A14) into the $\bar{b}$ frame and eliminating vectors yields the desired result,

$$\begin{bmatrix} \omega_{g_{o_1}} \\ \omega_{g_{o_2}} \\ \omega_{g_{o_3}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi_o & s\phi_o \\ 0 & -s\phi_o & c\phi_o \end{bmatrix} \begin{bmatrix} \omega_{b_1} + \dot{\theta}_o \\ \omega_{b_2} \\ \omega_{b_3} \end{bmatrix} . \quad \text{(A15)}$$

The dynamics of the gyro can be defined by expressing the momentum of the gyro in both the inner gimbal frame and the outer gimbal frame. The three following assumptions will be employed in deriving these equations:

1. The rotor revolves at constant speed ($\ddot{\psi}_r = 0$).
2. Only momentum terms containing $\dot{\psi}_r$ are appreciable (i.e. the momentum contribution of the gimbals is negligible).

3. The gyro rotor has a symmetrical mass distribution. Consider the inner gimbal dynamics,

$$\bar{H}_{rotor} = \bar{\bar{I}} \cdot \bar{\omega}_r \quad . \tag{A16}$$

Expanding equation (A16) and noting that $\bar{r}_3$ is the spin axis of the rotor gives

$$\bar{H}_{rotor} = \begin{bmatrix} \bar{r}_1 & \bar{r}_2 & \bar{r}_3 \end{bmatrix} \begin{bmatrix} I_{3_r} & 0 & 0 \\ 0 & I_{3_r} & 0 \\ 0 & 0 & I_{3_r} \end{bmatrix}$$

$$\cdot \quad \begin{bmatrix} \bar{r}_1 & \bar{r}_2 & \bar{r}_3 \end{bmatrix} \begin{bmatrix} \omega_{1_r} \\ \omega_{2_r} \\ \omega_{3_r} \end{bmatrix} . \tag{A17}$$

In transforming equation (A17) into the $\bar{g}_i$ frame, the rotor's inertia does not change due to the symmetry. Substituting equation (A7) into (A17) and noting symmetry yields

$$\bar{H}_{rotor} = \begin{bmatrix} \bar{g}_{i_1} & \bar{g}_{i_2} & \bar{g}_{i_3} \end{bmatrix} \begin{bmatrix} I_{3_r} & 0 & 0 \\ 0 & I_{3_r} & 0 \\ 0 & 0 & I_{3_r} \end{bmatrix} \begin{bmatrix} \bar{g}_{i_1} \\ \bar{g}_{i_2} \\ \bar{g}_{i_3} \end{bmatrix}$$

$$\cdot \begin{bmatrix} \bar{g}_{i_1} & \bar{g}_{i_2} & \bar{g}_{i_3} \end{bmatrix} \begin{bmatrix} \omega_{g_{i_1}} \\ \omega_{g_{i_2}} \\ \omega_{g_{i_3}} + \dot{\psi}_r \end{bmatrix} . \qquad \text{(A18)}$$

Expanding (A18) gives

$$\bar{H}_{rotor} = I_{3_r}\, \omega_{g_{i_1}} \bar{g}_{i_1} + I_{3_r}\, \omega_{g_{i_2}} \bar{g}_{i_2} + I_{3_r} \left(\omega_{g_{i_3}} + \dot{\psi}_r \right) \bar{g}_{i_3} . \qquad \text{(A19)}$$

The time derivative of equation (A19) is

$$\dot{\bar{H}}_{rotor} = \Sigma \text{ external } \tau \rightarrow \bar{g}_{i_2} \text{ axis} ,$$

hence,

$$\dot{\bar{H}}_{rotor} = I_{3_r} \dot{\omega}_{g_{i_1}} \bar{g}_{i_1} + I_{3_r} \dot{\omega}_{g_{i_2}} \bar{g}_{i_2} + I_{3_r} \left(\dot{\omega}_{g_{i_3}} + \ddot{\psi}_r \right) \bar{g}_{i_3}$$

$$+ \begin{vmatrix} \bar{g}_{i_1} & \bar{g}_{i_2} & \bar{g}_{i_3} \\ \omega_{g_{i_1}} & \omega_{g_{i_2}} & \omega_{g_{i_3}} \\ I_{3_r} \omega_{g_{i_1}} & I_{3_r} \omega_{g_{i_2}} & I_{3_r} \left(\omega_{g_{i_3}} + \dot{\psi}_r \right) \end{vmatrix} . \qquad \text{(A20)}$$

Since the $\bar{g}_{i_1}$ and $\bar{g}_{i_3}$ axes have no gimballed freedom, only the $\bar{g}_{i_2}$ component need be considered. Applying assumptions (1) and (2) to simplify equation (A20) gives

$$-h\omega_{g_{i_1}}\bar{g}_{i_2} = \sum \text{external } \tau \rightarrow \bar{g}_{i_2} \text{ axis}, \tag{A21}$$

where

$$h = I_{3_r}\dot{\psi}_r \; .$$

Now consider the outer gimbal dynamics. Substituting equation (A10) into equation (A20) gives

$$\bar{H}_{rotor} = \left[\bar{g}_{o_1}, \bar{g}_{o_2}\, \bar{g}_{o_3}\right] \begin{bmatrix} I_{3_r} & 0 & 0 \\ 0 & I_{3_r} & 0 \\ 0 & 0 & I_{3_r} \end{bmatrix} \begin{bmatrix} \bar{g}_{o_1} \\ \bar{g}_{o_2} \\ \bar{g}_{o_3} \end{bmatrix}$$

$$\cdot \left[\bar{g}_{o_1}\, \bar{g}_{o_2}\, \bar{g}_{o_3}\right] \begin{bmatrix} \omega_{g_{o_1}} + \dot{\psi}_r s\theta_i \\ \omega_{g_{o_2}} + \dot{\theta}_i \\ \omega_{g_{o_3}} + \dot{\psi}_r c\theta_i \end{bmatrix}, \tag{A22}$$

upon expanding

$$\bar{H}_{rotor} = I_{3_r}(\omega_{g_{o_1}} + \dot{\psi}_r s\theta_i)\bar{g}_{o_1} + I_{3_r}(\omega_{g_{o_2}} + \dot{\theta}_i)\bar{g}_{o_2}$$

$$+ I_{3_r}(\omega_{g_{o_3}} + \dot{\psi}_r c\theta_i)\bar{g}_{o_3} \; . \tag{A23}$$

Taking the derivative of equation (A23) with respect to time yields

$$\dot{\bar{H}}_{rotor} = \Sigma \text{ external } \tau \rightarrow \bar{g}_{o_1} \text{ axis ,}$$

or

$$\dot{\bar{H}}_{rotor} = I_{3_r}(\dot{\omega}_{g_{o_1}} + \dot{\psi}_r \dot{\theta}_i c\theta_i)\,\bar{g}_{o_1} + I_{3_r}(\dot{\omega}_{g_{o_2}} + \ddot{\theta}_i)\,\bar{g}_{o_2} + I_{3_r}(\dot{\omega}_{g_{o_3}} - \dot{\psi}_r \dot{\theta}_i s\theta_i)\,\bar{g}_{o_3} \tag{A24}$$

$$+ \begin{vmatrix} \bar{g}_{o_1} & \bar{g}_{o_2} & \bar{g}_{o_3} \\ \omega_{g_{o_1}} & \omega_{g_{o_2}} & \omega_{g_{o_3}} \\ I_{3_r}(\omega_{g_{o_1}} + \dot{\psi}_r s\theta_i) & I_{3_r}(\omega_{g_{o_2}} + \dot{\theta}_i) & I_{3_r}(\omega_{g_{o_3}} + \dot{\psi}_r c\theta) \end{vmatrix} .$$

Applying assumption (2) and noting that only the $\bar{g}_{o_1}$ axis component is of concern, this being the output axis of the gimbal, yields

$$h c\theta_i (\dot{\theta}_i + \omega_{g_{o_2}}) = \Sigma \text{ external } \tau \rightarrow \bar{g}_{o_1} \text{ axis .} \tag{A25}$$

APPENDIX B

DERIVATION OF INERTIAL ANGULAR VELOCITY WITH RESPECT TO THE PLANET FRAME

B1. Derivation of $\bar{\omega}_b$ as a Function of $\bar{\omega}_s$

The pitch, θ_b; roll, ϕ_b, and yaw, ψ_b; of the body is an Euler sequence that can be used to define the orientation of the $\bar{b}$ frame with respect to the $\bar{s}$ frame. The rotation sequence is arbitrary; however, it should be chosen to simplify mathematical manipulations. Yaw is the first rotation since steering will be the main angular movement of the vehicle. The second rotation in the sequence is pitch, and the last is roll.

In order to orient the $\bar{b}$ frame with respect to the $\bar{s}$ frame, two intermediate frames, $\bar{e}$ and $\bar{f}$, are introduced (see Figure 2). As in Appendix A, two coordinate systems can be related using a matrix frame transformation and the corresponding inter-frame velocity relationship. Therefore, relating the $\bar{e}$ and $\bar{s}$ frames,

$$\begin{bmatrix} \bar{e}_1 \\ \bar{e}_2 \\ \bar{e}_3 \end{bmatrix} = \begin{bmatrix} c\psi_b & s\psi_b & 0 \\ -s\psi_b & c\psi_b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{s}_1 \\ \bar{s}_2 \\ \bar{s}_3 \end{bmatrix} , \tag{B1}$$

and

$$\bar{\omega}_e = \bar{\omega}_s + \dot{\psi}_b \bar{s}_3 . \tag{B2}$$

Relating the $\bar{f}$ and $\bar{e}$ frames,

$$\begin{bmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \bar{f}_3 \end{bmatrix} = \begin{bmatrix} c\theta_b & 0 & -s\theta_b \\ 0 & 1 & 0 \\ s\theta_b & 0 & c\theta_b \end{bmatrix} \begin{bmatrix} \bar{e}_1 \\ \bar{e}_2 \\ \bar{e}_3 \end{bmatrix} , \tag{B3}$$

and

$$\bar{\omega}_f = \bar{\omega}_e + \dot{\theta}_b \bar{e}_2 \quad . \tag{B4}$$

Finally, the equations for the remaining two frames are

$$\begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \bar{b}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi_b & s\phi_b \\ 0 & -s\phi_b & c\phi_b \end{bmatrix} \begin{bmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \bar{f}_3 \end{bmatrix} , \tag{B5}$$

and

$$\bar{\omega}_b = \bar{\omega}_f + \dot{\phi}_b \bar{f}_1 \quad . \tag{B6}$$

Expanding equations (B2), (B4), and (B6) gives

$$\begin{bmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \end{bmatrix} \begin{bmatrix} \omega_{e_1} \\ \omega_{e_2} \\ \omega_{e_3} \end{bmatrix} = \begin{bmatrix} \bar{s}_1 & \bar{s}_2 & \bar{s}_3 \end{bmatrix} \begin{bmatrix} \omega_{s_1} \\ \omega_{s_2} \\ \omega_{s_3} \end{bmatrix} + \begin{bmatrix} \bar{s}_1 & \bar{s}_2 & \bar{s}_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\psi}_b \end{bmatrix} , \tag{B7}$$

$$\begin{bmatrix} \bar{f}_1 & \bar{f}_2 & \bar{f}_3 \end{bmatrix} \begin{bmatrix} \omega_{f_1} \\ \omega_{f_2} \\ \omega_{f_3} \end{bmatrix} = \begin{bmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \end{bmatrix} \begin{bmatrix} \omega_{e_1} \\ \omega_{e_2} \\ \omega_{e_3} \end{bmatrix} + \begin{bmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\theta}_b \\ 0 \end{bmatrix} , \tag{B8}$$

and

$$\begin{bmatrix}\bar{b}_1 & \bar{b}_2 & \bar{b}_3\end{bmatrix}\begin{bmatrix}\omega_{b_1}\\ \omega_{b_2}\\ \omega_{b_3}\end{bmatrix} = \begin{bmatrix}\bar{f}_1 & \bar{f}_2 & \bar{f}_3\end{bmatrix}\begin{bmatrix}\omega_{f_1}\\ \omega_{f_2}\\ \omega_{f_3}\end{bmatrix} + \begin{bmatrix}\bar{f}_1 & \bar{f}_2 & \bar{f}_3\end{bmatrix}\begin{bmatrix}\dot{\phi}_b\\ 0\\ 0\end{bmatrix}, \quad \text{(B9)}$$

respectively.

Substituting equations (B3) and (B8) into equation (B9) yields

$$\begin{bmatrix}\bar{b}_1 & \bar{b}_2 & \bar{b}_3\end{bmatrix}\begin{bmatrix}\omega_{b_1}\\ \omega_{b_2}\\ \omega_{b_3}\end{bmatrix} = \begin{bmatrix}\bar{e}_1 & \bar{e}_2 & \bar{e}_3\end{bmatrix}\begin{bmatrix}\omega_{e_1}\\ \omega_{e_2}\\ \omega_{e_3}\end{bmatrix} + \begin{bmatrix}\bar{e}_1 & \bar{e}_2 & \bar{e}_3\end{bmatrix}\begin{bmatrix}0\\ \dot{\theta}_b\\ 0\end{bmatrix}$$

$$+ \begin{bmatrix}\bar{e}_1 & \bar{e}_2 & \bar{e}_3\end{bmatrix}\begin{bmatrix}c\theta_b & 0 & s\theta_b\\ 0 & 1 & 0\\ -s\theta_b & 0 & c\theta_b\end{bmatrix}\begin{bmatrix}\dot{\psi}_b\\ 0\\ 0\end{bmatrix}. \quad \text{(B10)}$$

Substituting equations (B1) and (B7) into equation (B10) produces

$$\begin{bmatrix}\bar{b}_1 & \bar{b}_2 & \bar{b}_3\end{bmatrix}\begin{bmatrix}\omega_{b_1}\\ \omega_{b_2}\\ \omega_{b_3}\end{bmatrix} = \begin{bmatrix}\bar{s}_1 & \bar{s}_2 & \bar{s}_3\end{bmatrix}\begin{bmatrix}\omega_{s_1}\\ \omega_{s_2}\\ \omega_{s_3}\end{bmatrix} + \begin{bmatrix}\bar{s}_1 & \bar{s}_2 & \bar{s}_3\end{bmatrix}\begin{bmatrix}0\\ 0\\ \dot{\psi}_b\end{bmatrix}$$

$$+ \begin{bmatrix}\bar{s}_1 & \bar{s}_2 & \bar{s}_3\end{bmatrix}\begin{bmatrix}c\psi_b & -s\psi_b & 0\\ s\psi_b & c\psi_b & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\ \dot{\theta}_b\\ 0\end{bmatrix}$$

$$+ \begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix} \begin{bmatrix} c\psi_b & -s\psi_b & 0 \\ s\psi_b & c\psi_b & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\cdot \begin{bmatrix} c\theta_b & 0 & s\theta_b \\ 0 & 1 & 0 \\ -s\theta_b & 0 & c\theta_b \end{bmatrix} \begin{bmatrix} \dot{\phi}_b \\ 0 \\ 0 \end{bmatrix} , \qquad \text{(B11)}$$

or

$$\begin{bmatrix} \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \end{bmatrix} \begin{bmatrix} \omega_{b_1} \\ \omega_{b_2} \\ \omega_{b_3} \end{bmatrix} = \begin{bmatrix} \bar{s}_1 & \bar{s}_2 & \bar{s}_3 \end{bmatrix} \begin{bmatrix} \omega_{s_1} \\ \omega_{s_2} \\ \omega_{s_3} \end{bmatrix}$$

$$\cdot \begin{bmatrix} \bar{s}_1 & \bar{s}_2 & \bar{s}_3 \end{bmatrix} \begin{bmatrix} -\dot{\theta}_b s\psi_b + \dot{\phi}_b c\psi_b c\theta_b \\ \dot{\theta}_b c\psi_b + \dot{\phi}_b s\psi_b c\theta_b \\ \dot{\psi}_b - \dot{\phi}_b s\theta_b \end{bmatrix} . \qquad \text{(B12)}$$

Solving for $\bar{s}$ as a function of $\bar{b}$ in equations (B1), (B3), and (B5), and using this result in equation (B12) yields

$$\begin{bmatrix} \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \end{bmatrix} \begin{bmatrix} \omega_{b_1} \\ \omega_{b_2} \\ \omega_{b_3} \end{bmatrix} = \begin{bmatrix} \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \end{bmatrix}$$

$$\cdot \begin{bmatrix} c\theta_b c\psi_b & c\theta_b s\psi_b & -s\theta_b \\ s\phi_b s\theta_b c\psi_b - c\phi_b s\psi_b & s\psi_b s\phi_b s\theta_b + c\phi_b c\psi_b & s\phi_b c\theta_b \\ c\phi_b s\theta_b c\psi_b + s\phi_b s\psi_b & c\phi_b s\theta_b s\psi_b - s\phi_b c\psi_b & c\phi_b c\theta_b \end{bmatrix}$$

$$\cdot \begin{bmatrix} \omega_{s_1} - \dot{\theta}_b s\psi_b + \dot{\phi}_b c\psi_b c\theta_b \\ \omega_{s_2} + \dot{\theta}_b c\psi_b + \dot{\phi}_b s\psi_b c\theta_b \\ \omega_{s_3} + \dot{\psi}_b - \dot{\phi}_b s\theta \end{bmatrix} \cdot \qquad \text{(B13)}$$

Eliminating the vectors and simplifying equation (B13) gives

$$\begin{bmatrix} \omega_{b_1} \\ \omega_{b_2} \\ \omega_{b_3} \end{bmatrix} = \begin{bmatrix} \dot{\phi}_b - \dot{\psi}_b s\theta_b \\ \dot{\phi}_b c\phi_b + \dot{\psi}_b s\phi_b c\theta_b \\ \dot{\psi}_b c\phi_b c\theta_b - \dot{\theta}_b s\phi_b \end{bmatrix} + M_{b/s} \begin{bmatrix} \omega_{s_1} \\ \omega_{s_2} \\ \omega_{s_3} \end{bmatrix} , \qquad \text{(B14)}$$

where

$$M_{b/s} = \begin{bmatrix} c\theta_b c\psi_b & c\theta_b s\psi_b & -s\theta_b \\ s\phi_b s\theta_b c\psi_b - c\phi_b s\psi_b & s\phi_b s\theta_b s\psi_b + c\phi_b c\psi_b & s\phi_b c\theta_b \\ c\phi_b s\theta_b c\psi_b + s\phi_b s\psi_b & c\phi_b s\theta_b s\psi_b - s\phi_b c\psi_b & c\phi_b c\theta_b \end{bmatrix} .$$

B2. Derivation of $\bar{\omega}_s$ as a Function of $\bar{\omega}_p$

Vectors in the $\bar{s}$ frame can be expressed in the $\bar{p}$ frame through a sequence of two Euler rotations. The first one is chosen to be a longitude (γ_L) rotation; and the second, a latitude (θ_L) rotation (see Figure 3). The vector $\bar{s}_1$ is tangent to the great circle whose plane is located γ_L from $\bar{p}_1$; and $\bar{s}_2$ is tangent to the minor circle defined by the declination, θ_L, from the pole; while $\bar{s}_3$ lies along the local vertical.

As done previously, an intermediate frame, $\bar{y}$, is introduced such that

$$\begin{bmatrix} \bar{s}_1 \\ \bar{s}_2 \\ \bar{s}_3 \end{bmatrix} = \begin{bmatrix} c\theta_L & 0 & -s\theta_L \\ 0 & 1 & 0 \\ s\theta_L & 0 & c\theta_L \end{bmatrix} \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \end{bmatrix} , \tag{B15}$$

and

$$\begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \end{bmatrix} = \begin{bmatrix} c\gamma_L & s\gamma_L & 0 \\ -s\gamma_L & c\gamma_L & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{p}_1 \\ \bar{p}_2 \\ \bar{p}_3 \end{bmatrix} . \tag{B16}$$

The angular velocity relationships corresponding to equations (B15) and (B16) are

$$\bar{\omega}_s = \bar{\omega}_y + \dot{\theta}_L \bar{y}_2 , \tag{B17}$$

and

$$\bar{\omega}_y = \bar{\omega}_p + \dot{\gamma}_L \bar{p}_3 , \tag{B18}$$

respectively.

Expanding equations (B17) and (B18) yields

$$\begin{bmatrix}\bar{s}_1 & \bar{s}_2 & \bar{s}_3\end{bmatrix}\begin{bmatrix}\omega_{s_1}\\ \omega_{s_2}\\ \omega_{s_3}\end{bmatrix} = \begin{bmatrix}\bar{y}_1 & \bar{y}_2 & \bar{y}_3\end{bmatrix}\begin{bmatrix}\omega_{y_1}\\ \omega_{y_2}\\ \omega_{y_3}\end{bmatrix} + \begin{bmatrix}\bar{y}_1 & \bar{y}_2 & \bar{y}_3\end{bmatrix}\begin{bmatrix}0\\ \dot{\theta}_L\\ 0\end{bmatrix}, \quad \text{(B19)}$$

and

$$\begin{bmatrix}\bar{y}_1 & \bar{y}_2 & \bar{y}_3\end{bmatrix}\begin{bmatrix}\omega_{y_1}\\ \omega_{y_2}\\ \omega_{y_3}\end{bmatrix} = \begin{bmatrix}\bar{p}_1 & \bar{p}_2 & \bar{p}_3\end{bmatrix}\begin{bmatrix}\omega_{p_1}\\ \omega_{p_2}\\ \omega_{p_3}\end{bmatrix} + \begin{bmatrix}\bar{p}_1 & \bar{p}_2 & \bar{p}_3\end{bmatrix}\begin{bmatrix}0\\ 0\\ \dot{\psi}_L\end{bmatrix}. \quad \text{(B20)}$$

Substituting equations (B16) and (B20) into equation (B19) gives

$$\begin{bmatrix}\bar{s}_1 & \bar{s}_2 & \bar{s}_3\end{bmatrix}\begin{bmatrix}\omega_{s_1}\\ \omega_{s_2}\\ \omega_{s_3}\end{bmatrix} = \begin{bmatrix}\bar{p}_1 & \bar{p}_2 & \bar{p}_3\end{bmatrix}\begin{bmatrix}\omega_{p_1}\\ \omega_{p_2}\\ \omega_{p_3}\end{bmatrix} + \begin{bmatrix}\bar{p}_1 & \bar{p}_2 & \bar{p}_3\end{bmatrix}\begin{bmatrix}0\\ 0\\ \dot{\psi}_L\end{bmatrix}$$

$$+ \begin{bmatrix}\bar{p}_1 & \bar{p}_2 & \bar{p}_3\end{bmatrix}\begin{bmatrix}c\psi_L & -s\psi_L & 0\\ s\psi_L & c\psi_L & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\ \dot{\theta}_L\\ 0\end{bmatrix}, \quad \text{(B21)}$$

or

$$\begin{bmatrix}\bar{s}_1 & \bar{s}_2 & \bar{s}_3\end{bmatrix}\begin{bmatrix}\omega_{s_1}\\ \omega_{s_2}\\ \omega_{s_3}\end{bmatrix} = \begin{bmatrix}\bar{p}_1 & \bar{p}_2 & \bar{p}_3\end{bmatrix}\begin{bmatrix}\omega_{p_1} - \dot{\theta}_L s\psi_L\\ \omega_{p_2} + \dot{\theta}_L c\psi_L\\ \omega_{p_3} + \dot{\psi}_L\end{bmatrix}. \quad \text{(B22)}$$

Solving for $\bar{s}$ as a function of $\bar{p}$ in equations (B15) and (B16), and using this result in equation (B22) produces

$$\begin{bmatrix} \bar{s}_1 & \bar{s}_2 & \bar{s}_3 \end{bmatrix} \begin{bmatrix} \omega_{s_1} \\ \omega_{s_2} \\ \omega_{s_3} \end{bmatrix} = \begin{bmatrix} \bar{s}_1 & \bar{s}_2 & \bar{s}_3 \end{bmatrix} \begin{bmatrix} c\theta_L\, c\psi_L & c\theta_L\, s\psi_L & -s\theta_L \\ -s\psi_L & c\psi_L & 0 \\ s\theta_L\, c\psi_L & s\theta_L\, s\psi_L & c\theta_L \end{bmatrix} \bullet \begin{bmatrix} \omega_{p_1} - \dot{\theta}_L\, s\psi_L \\ \omega_{p_2} + \dot{\theta}_L\, c\psi_L \\ \omega_{p_3} + \dot{\psi}_L \end{bmatrix} . \tag{B23}$$

Eliminating vectors and simplifying the above result gives

$$\begin{bmatrix} \omega_{s_1} \\ \omega_{s_2} \\ \omega_{s_3} \end{bmatrix} = \begin{bmatrix} -\dot{\psi}_K\, s\theta_L - \dot{\psi}_L\, s\theta_L \\ \dot{\theta}_L \\ \dot{\psi}_K\, c\theta_L + \dot{\psi}_L\, c\theta_L \end{bmatrix} . \tag{B24}$$

APPENDIX C

Derivation of Translational Equations of Motion of the Rover

The location of the rover with respect to the $\bar{p}_1$, $\bar{p}_2$, $\bar{p}_3$ coordinate system is designated by x, y, and z, respectively. (See figure 3) The position of the vehicle is defined to be

$$\bar{R} = x\bar{p}_1 + y\bar{p}_2 + z\bar{p}_3 \quad , \tag{C1}$$

which leads to the scalar constraint equation of

$$R^2 = x^2 + y^2 + z^2 \quad , \tag{C2}$$

where R is the radius of Mars and is assumed constant. (The height of any hill that the rover traverses is considered negligible with respect to R.)

The first and second time derivatives of equation (C2) are

$$x\dot{x} + y\dot{y} + z\dot{z} = 0 \quad , \tag{C3}$$

and

$$x\ddot{x} + \dot{x}^2 + y\ddot{y} + \dot{y}^2 + z\ddot{z} + \dot{z}^2 = 0 \quad , \tag{C4}$$

respectively. Equations (C3) and (C4) are used in conjunction with the velocity sensors and will be treated in greater detail later.

The velocity of the vehicle is the time derivative of equation (C1) with respect to inertial space

$$\bar{v} = \dot{\bar{R}} \quad , \tag{C5}$$

or

$$\bar{v} = \dot{x}\bar{p}_1 + \dot{y}\bar{p}_2 + \dot{z}\bar{p}_3 + \begin{vmatrix} \bar{p}_1 & \bar{p}_2 & \bar{p}_3 \\ \omega_{p_1} & \omega_{p_2} & \omega_{p_3} \\ x & y & z \end{vmatrix} . \quad \text{(C6)}$$

Since the $\bar{p}$ frame has only one degree of freedom with respect to inertial space,

$$\begin{bmatrix} \omega_{p_1} \\ \omega_{p_2} \\ \omega_{p_3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\gamma}_k \end{bmatrix} , \quad \text{(C7)}$$

where $\dot{\gamma}_k$ is the rotational rate of the planet.

Substituting equation (C7) into equation (C6) and simplifying yields

$$\bar{v} = (\dot{x} - y\dot{\gamma}_k)\bar{p}_1 + (\dot{y} + x\dot{\gamma}_k)\bar{p}_2 + \dot{z}\bar{p}_3 . \quad \text{(C8)}$$

The time derivative of equation (C8) with respect to inertial space yields the acceleration of the vehicle,

$$\bar{a} = (\ddot{x} - \dot{y}\dot{\gamma}_k)\bar{p}_1 + (\ddot{y} + \dot{x}\dot{\gamma}_k)\bar{p}_2 + \ddot{z}\bar{p}_3$$

$$+ \begin{vmatrix} p_1 & p_2 & p_3 \\ \omega_{p_1} & \omega_{p_2} & \omega_{p_3} \\ \dot{x} - y\dot{\gamma}_k & \dot{y} - x\dot{\gamma}_k & \dot{z} \end{vmatrix} . \quad \text{(C9)}$$

Substituting equation (C7) into equation (C9) and collecting terms gives the total inertial acceleration of the rover.

$$\bar{a} = \left[\ddot{x} - \dot{\psi}_k(2\dot{y} + x\dot{\psi}_k)\right]\bar{p}_1 + \left[\dot{\psi}_k(2\dot{x} - y\dot{\psi}_k) + \ddot{y}\right]\bar{p}_2 + \ddot{z}\,\bar{p}_3 \, . \tag{C10}$$

Newton's second law of motion can now be used to define the translation of the vehicle,

$$\bar{F} = M\bar{a} \, , \tag{C11}$$

where M is the total mass of the craft. The three forces acting on the vehicle are: the force due to gravity, $\bar{F}_g$; the reaction force of the planet on the vehicle, $\bar{F}_r$; and the internal driving force of the rover itself, $\bar{F}_v$, where it is assumed that $\bar{F}_v$ acts along the $\bar{b}_1$ axis of the vehicle (see Figure 4). Therefore, upon expanding equation (C11) one obtains

$$\bar{F}_v + \bar{F}_g + \bar{F}_r = M\Big\{\left[\ddot{x} - \dot{\psi}_k(2\dot{y} + x\dot{\psi}_k)\right]\bar{p}_1 + \left[\ddot{y} + \dot{\psi}_k(2\dot{x} - y\dot{\psi}_k)\right]\bar{p}_2 + \ddot{z}\,\bar{p}_3\Big\} \, . \tag{C12}$$

When the vehicle is at rest, $\bar{F}_v = 0$, and x, y, and z are constant (all time derivatives are also zero). Using this information in equation (C12) gives

$$\bar{F}_g + \bar{F}_r = M\left(-x\dot{\psi}_k^2\bar{p}_1 - y\dot{\psi}_k^2\bar{p}_2\right) \, . \tag{C13}$$

Substituting equation (C13) into (C12) produces

$$\bar{F}_v = M\left[(\ddot{x} - 2\dot{\psi}_k\dot{y})\bar{p}_1 + (\ddot{y} - 2\dot{\psi}_k\dot{x})\bar{p}_2 + \ddot{z}\,\bar{p}_3\right] \, . \tag{C14}$$

In order to use equation (C14) in later work, it is necessary to express both sides of the relation in the same coordinate system. As stated previously,

$$\bar{F}_v = F_v \bar{b}_1 \tag{C15}$$

Since $\bar{F}_v$ acts in only the $\bar{b}_1$ direction, the mathematics is simplified by expressing equation (C14) in the $\bar{p}$ frame.

Using equations (B1), (B3), and (B5), the matrix transformation relating the $\bar{b}$ and $\bar{s}$ frame results.

$$\begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \bar{b}_3 \end{bmatrix} = \begin{bmatrix} c\theta_b c\psi_b & c\theta_b s\psi_b & -s\theta_b \\ s\phi_b s\theta_b c\psi_b - c\phi_b s\psi_b & s\phi_b s\theta_b s\psi_b + c\phi_b c\psi_b & s\phi_b c\theta_b \\ c\phi_b s\theta_b c\psi_b + s\phi_b s\psi_b & c\phi_b s\theta_b s\psi_b - s\phi_b c\psi_b & c\phi_b c\psi_b \end{bmatrix} \begin{bmatrix} \bar{s}_1 \\ \bar{s}_2 \\ \bar{s}_3 \end{bmatrix} . \tag{C16}$$

Using equations (B15) and (B16),

$$\begin{bmatrix} \bar{s}_1 \\ \bar{s}_2 \\ \bar{s}_3 \end{bmatrix} = \begin{bmatrix} c\theta_2 c\psi_2 & c\theta_2 & -s\theta_2 \\ -s\psi_2 & c\psi_2 & 0 \\ s\theta_2 c\psi_2 & s\theta_2 s\psi_2 & c\theta_2 \end{bmatrix} \begin{bmatrix} \bar{p}_1 \\ \bar{p}_2 \\ \bar{p}_3 \end{bmatrix} . \tag{C17}$$

Equations (C16) and (C17) can now be used to solve for $\bar{b}$ as a function of $\bar{p}$.

$$\begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \bar{b}_3 \end{bmatrix} = M_{b/p} \begin{bmatrix} \bar{p}_1 \\ \bar{p}_2 \\ \bar{p}_3 \end{bmatrix} , \qquad \text{(C18)}$$

where

$$M_{b/p}(1,1) = c\theta_b c\psi_b c\theta_L c\psi_L - c\theta_b s\psi_b s\psi_L - s\theta_b s\theta_L c\psi_L ,$$

$$M_{b/p}(1,2) = c\theta_b c\psi_b c\theta_L s\psi_L + c\theta_b s\psi_b c\psi_L - s\theta_b s\theta_L s\psi_L ,$$

$$M_{b/p}(1,3) = -c\theta_b c\psi_b s\theta_L - s\theta_b c\theta_L ,$$

$$\begin{aligned} M_{b/p}(2,1) = {} & c\theta_L c\psi_L (s\theta_b s\phi_b c\psi_b - c\phi_b s\psi_b) \\ & - s\psi_L (s\phi_b s\theta_b s\psi_b + c\phi_b c\psi_b) + s\phi_b c\theta_b s\theta_L c\psi_L , \end{aligned}$$

$$\begin{aligned} M_{b/p}(2,2) = {} & c\theta_L s\psi_L (s\theta_b s\phi_b c\psi_b - c\phi_b s\psi_b) \\ & + c\psi_L (s\phi_b s\theta_b s\psi_b + c\phi_b c\psi_b) + s\phi_b c\theta_b s\theta_L s\psi_L , \end{aligned}$$

$$M_{b/p}(2,3) = -s\theta_L (s\theta_b s\phi_b c\psi_b - c\phi_b s\psi_b) + c\theta_L s\phi_b c\theta_b ,$$

$$\begin{aligned} M_{b/p}(3,1) = {} & c\theta_L c\psi_L (c\phi_b s\theta_b c\psi_b + s\phi_b s\psi_b) - s\psi_L \\ & \cdot (c\phi_b s\theta_b s\psi_b - s\phi_b c\psi_b) + s\theta_L c\psi_L c\phi_b c\theta_b , \end{aligned}$$

$$\begin{aligned} M_{b/p}(3,2) = {} & c\theta_L s\psi_L (c\phi_b s\theta_b c\psi_b + s\phi_b s\psi_b) \\ & + c\psi_L (c\phi_b s\theta_b s\psi_b - s\phi_b c\psi_b) + s\theta_L s\psi_L c\phi_b c\theta_b , \end{aligned}$$

and

$$M_{b/p}(3,3) = -s\theta_L (c\phi_b s\theta_b c\psi_b + s\phi_b s\psi_b) + c\theta_L c\phi_b c\theta_b .$$

Substituting equations (C15) and (C18) into equation (C14) produces

$$F_v\left[M_{b/p}(1,1)\bar{p}_1 + M_{b/p}(1,2)\bar{p}_2 + M_{b/p}(1,3)\bar{p}_3\right] = M\left[(\ddot{x} - 2\dot{\psi}_K\dot{y})\bar{p}_1 + (\ddot{y} - 2\dot{\psi}_K\dot{x})\bar{p}_2 + \ddot{z}\bar{p}_3\right] \quad \text{(C19)}$$

Simplifying equation (C19) and clearing of vectors yields and desired translation equation,

$$\frac{F_v}{M}\begin{bmatrix} c\theta_b c\psi_b c\theta_L c\psi_L - c\theta_b s\psi_b s\psi_L - s\theta_b s\theta_L c\psi_L \\ c\theta_b c\psi_b c\theta_L s\psi_L + c\theta_b s\psi_b c\psi_L - s\theta_b s\theta_L s\psi_L \\ -c\theta_b c\psi_b s\theta_L - s\theta_b c\theta_L \end{bmatrix} = \begin{bmatrix} \ddot{x} - 2\dot{\psi}_K\dot{y} \\ \ddot{y} - 2\dot{\psi}_K\dot{x} \\ \ddot{z} \end{bmatrix} . \quad \text{(C20)}$$

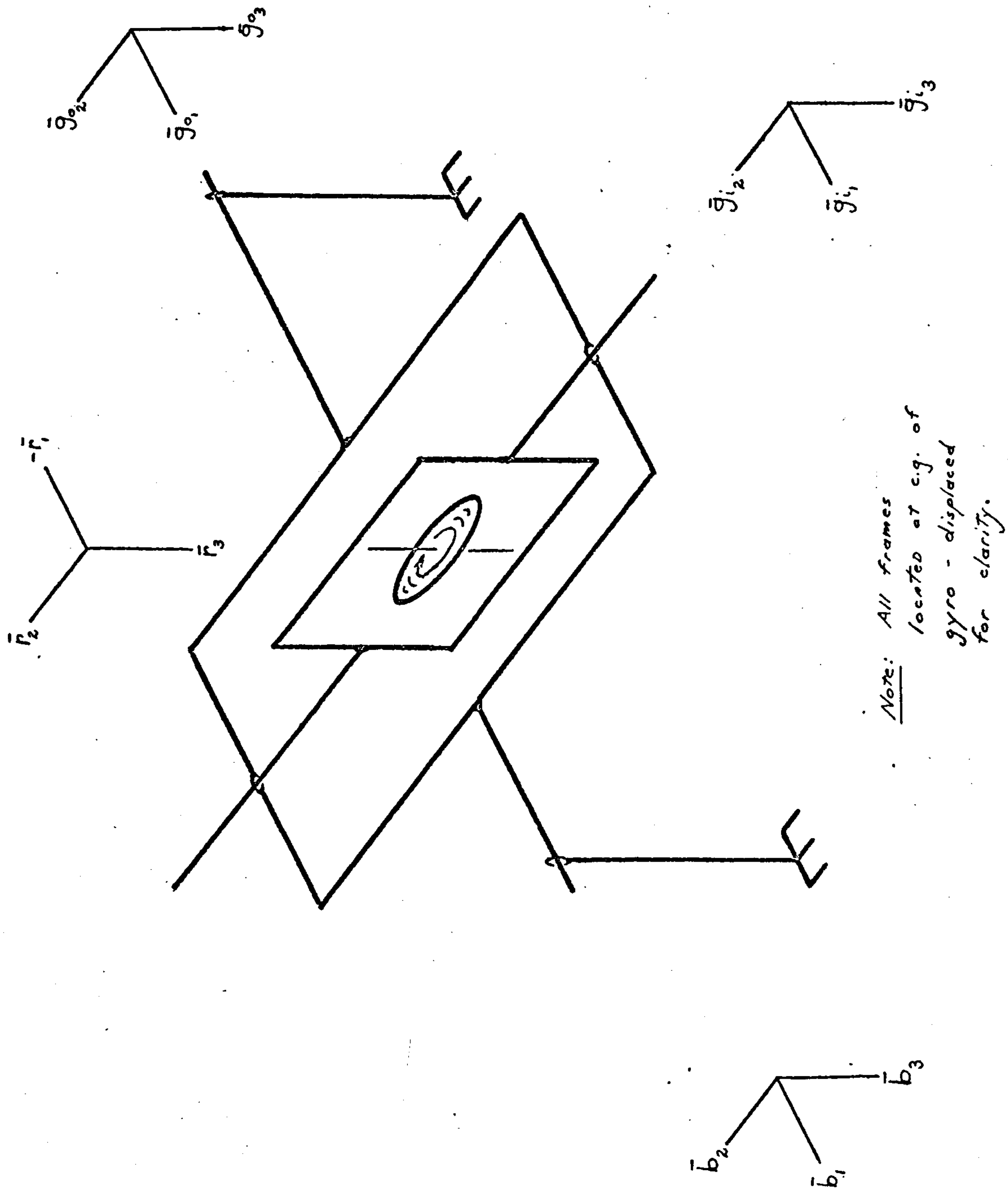


FIGURE 1. Two-Degrees-of-Freedom Gyroscope.

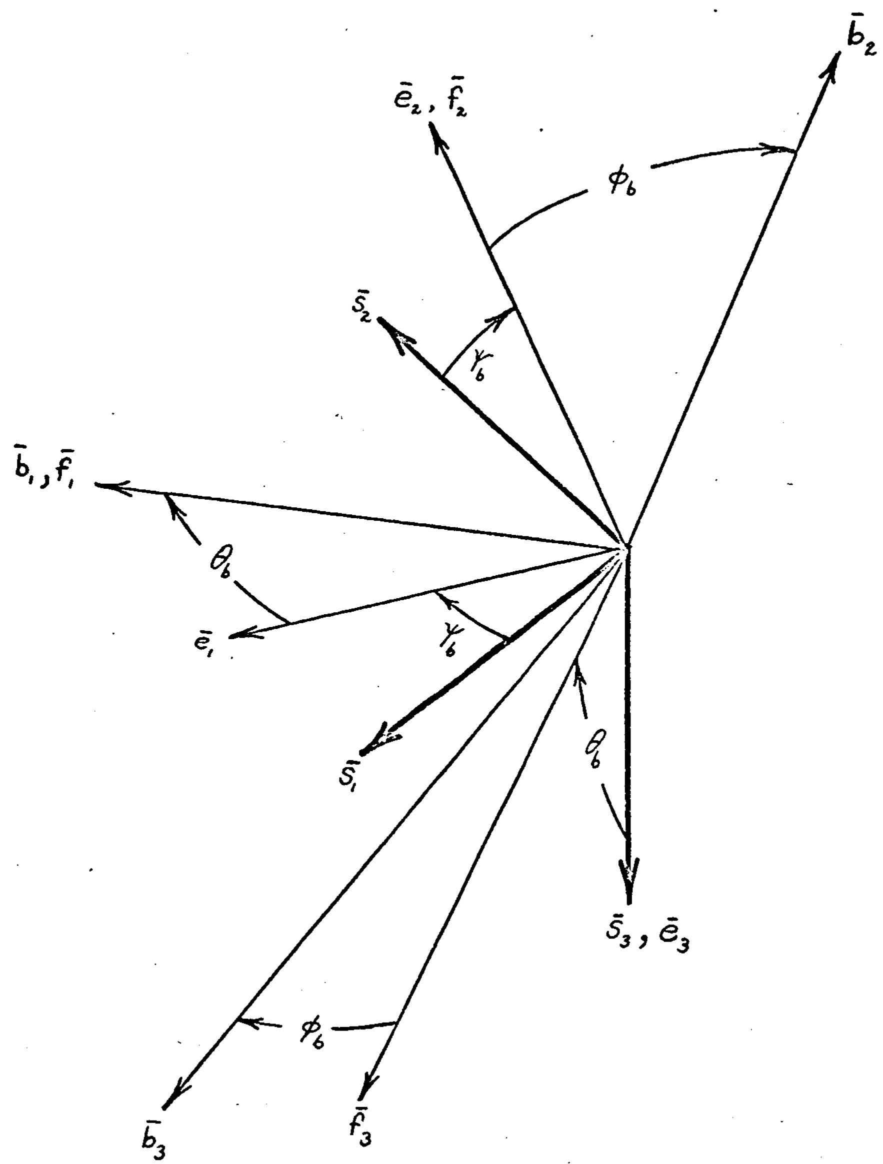
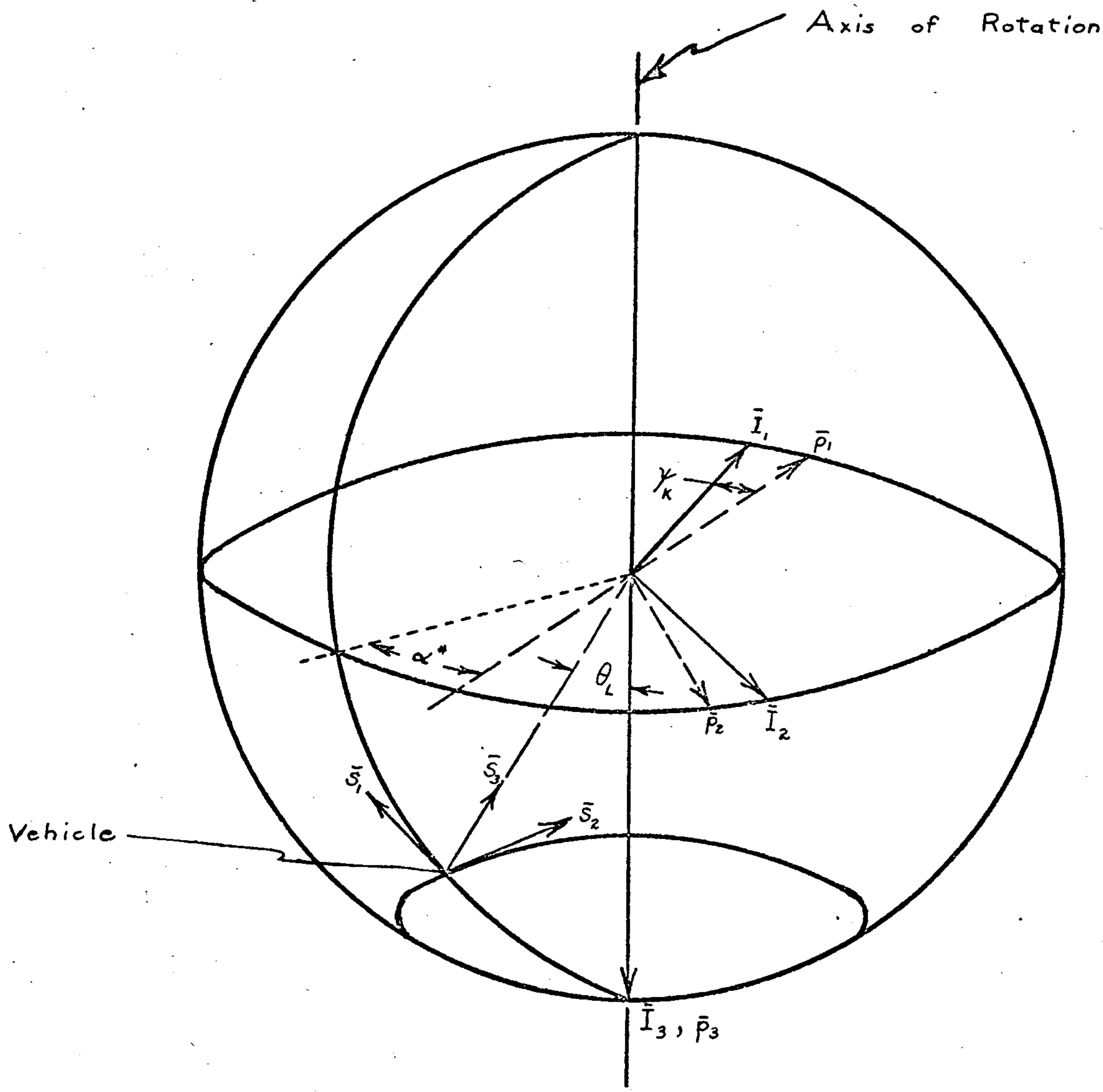


FIGURE 2. Rotational Scheme: $\bar{\bar{s}}$ frame to $\bar{b}$ frame.



$*\psi_L = \pi + \alpha$

FIGURE 3. Relationship of Martian Land Rover with Respect to Planetary Reference Frame

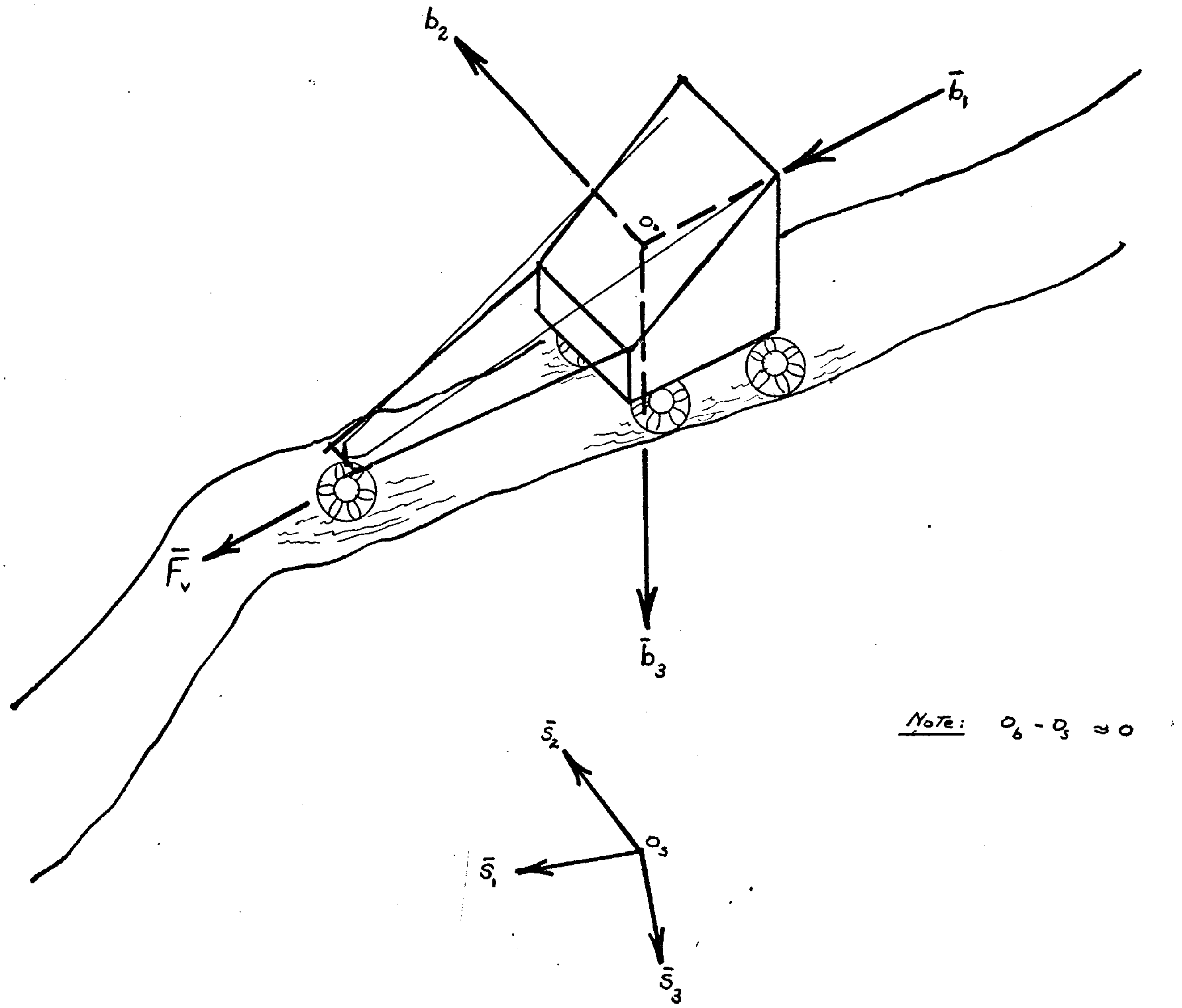


FIGURE 4. Relationship of Martian Land Rover with Respect to Surface